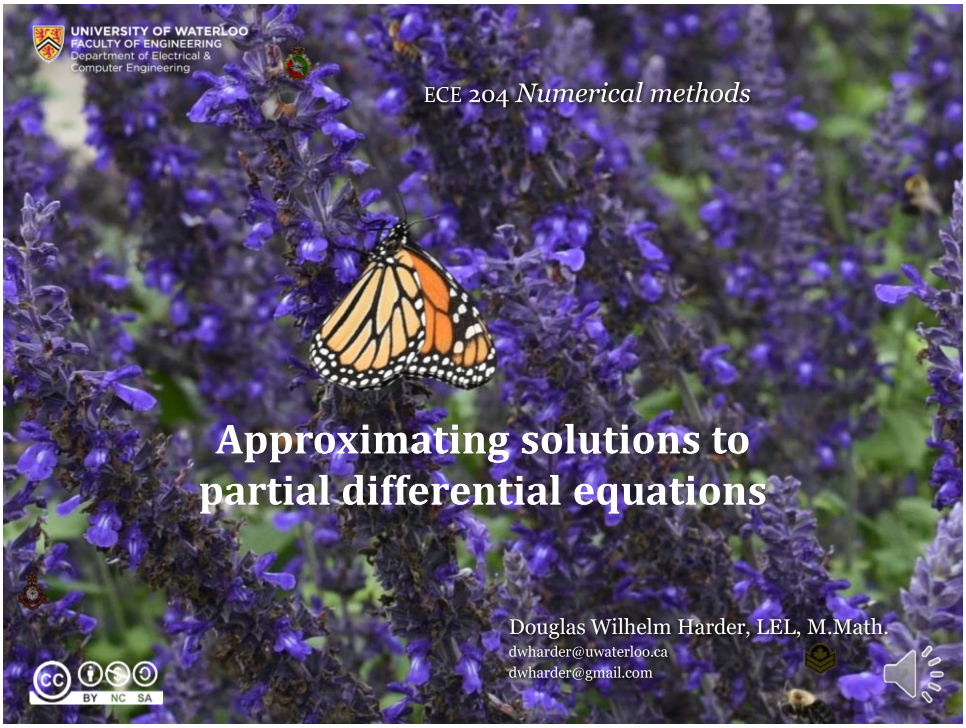

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Approximating solutions to partial differential equations

Introduction

- In this topic, we will
  - Review partial derivatives
  - Discuss partial derivatives with respect to functions of a vector variable
  - Describe the gradient and the Laplacian
  - List the upcoming topics

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Partial differential equations

- For a function of two or more variables,
  we can define a partial derivative
  - This assumes that the variables do not depend on each other

- The *total* derivative may be used if one variable depends on another
  - Refer to your calculus text books

Partial derivatives

- A function $u(t, x)$ of two variables has two partial derivatives

$$\frac{\partial}{\partial t}u(x,t) = u_t(x,t) \qquad \frac{\partial}{\partial x}u(x,t) = u_x(x,t)$$

  - The definition of a partial derivative is a limit:

$$\frac{\partial}{\partial t}u(x,t) = \lim_{\Delta t \to 0} \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

$$\frac{\partial}{\partial x}u(x,t) = \lim_{h \to 0} \frac{u(x+h,t) - u(x,t)}{h}$$

Approximating partial derivatives

- Consequently, we can approximate all these the same way

$$\frac{\partial}{\partial t}u(x,t) \approx \frac{u(x,t+\Delta t) - u(x,t-\Delta t)}{2\Delta t}$$

$$\frac{\partial^2}{\partial t^2}u(x,t) \approx \frac{u(x,t+\Delta t) - 2u(x,t) + u(x,t-\Delta t)}{(\Delta t)^2}$$

$$\frac{\partial}{\partial x}u(x,t) \approx \frac{u(x+h,t) - u(x-h,t)}{2h}$$

$$\frac{\partial^2}{\partial x^2}u(x,t) \approx \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}$$

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Functions of a vector variable

- Suppose $u(\mathbf{u})$ is a real-valued function of a vector variable
  - Suppose the vector is n-dimensional
- We can approximate a partial derivative with respect to each of the n variables

$$\frac{\partial}{\partial x_2}u(\mathbf{x}) \approx \frac{u\left(\begin{pmatrix} x_1 \\ x_2 + h \\ x_3 \end{pmatrix}\right) - u\left(\begin{pmatrix} x_1 \\ x_2 - h \\ x_3 \end{pmatrix}\right)}{2h}$$

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Functions of a vector variable

- Recall that $\mathbf{e}_k$ is often used to represent the k^{th} canonical basis vector

$$\mathbf{e}_1 \approx \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 \approx \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_3 \approx \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- Thus, we note that

$$\begin{pmatrix} x_1 \\ x_2 + h \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + h \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \mathbf{x} + h\mathbf{e}_2$$

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Approximating solutions to partial differential equations

Functions of a vector variable

- Thus, we have that

$$\frac{\partial}{\partial x_k} u(\mathbf{x}) \approx \frac{u(\mathbf{x} + h\mathbf{e}_k) - u(\mathbf{x} - h\mathbf{e}_k)}{2h}$$

$$\frac{\partial^2}{\partial x_k^2} u(\mathbf{x}) \approx \frac{u(\mathbf{x} + h\mathbf{e}_k) - 2u(\mathbf{x}) + u(\mathbf{x} - h\mathbf{e}_k)}{h^2}$$

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Functions of a vector variable

- We will deal with real-valued functions that are parameterized by both of time and space
  - Such a function will be denoted as $u(t, x)$ or $u(t, \mathbf{x})$

$$\frac{\partial}{\partial t}u(\mathbf{x},t) \approx \frac{u(\mathbf{x},t+\Delta t)-u(\mathbf{x},t-\Delta t)}{2\Delta t}$$

$$\frac{\partial^2}{\partial t^2}u(\mathbf{x},t) \approx \frac{u(\mathbf{x},t+\Delta t)-2u(\mathbf{x},t)+u(\mathbf{x},t-\Delta t)}{(\Delta t)^2}$$

$$\frac{\partial}{\partial x_k}u(\mathbf{x},t) \approx \frac{u(\mathbf{x}+h\mathbf{e}_k,t)-u(\mathbf{x}-h\mathbf{e}_k,t)}{2h}$$

$$\frac{\partial^2}{\partial x_k^2}u(\mathbf{x},t) \approx \frac{u(\mathbf{x}+h\mathbf{e}_k,t)-2u(\mathbf{x},t)+u(\mathbf{x}-h\mathbf{e}_k,t)}{h^2}$$

Approximating solutions to partial differential equations

Conditions

- Conditions depend on the variable:
  - The time variable will have initial conditions
  - The space variable or variables will have boundary conditions

- If the space variable is 1-dimensional, the boundary conditions must be specified on an interval $[a, b]$
- If the space variable is 2- or 3-dimensional, the boundary conditions must be specified on the boundary of 2- or 3-dimensional bounded region in space

The gradient and the Laplacian

- If $u(\mathbf{x})$ is a real-valued function of a vector variable, we will define the gradient as

$$\vec{\nabla} u(\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} u(\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1} u(\mathbf{x}) \\ \frac{\partial}{\partial x_2} u(\mathbf{x}) \\ \vdots \\ \frac{\partial}{\partial x_n} u(\mathbf{x}) \end{pmatrix}$$

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The gradient and the Laplacian

- If $u(\mathbf{x})$ is a real-valued function of a vector variable, we will define the Laplacian as

$$\vec{\nabla} \bullet \vec{\nabla} u(\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial}{\partial x_1} u(\mathbf{x}) \\ \frac{\partial}{\partial x_2} u(\mathbf{x}) \\ \vdots \\ \frac{\partial}{\partial x_n} u(\mathbf{x}) \end{pmatrix} = \frac{\partial^2}{\partial x_1^2} u(\mathbf{x}) + \frac{\partial^2}{\partial x_2^2} u(\mathbf{x}) + \cdots + \frac{\partial^2}{\partial x_n^2} u(\mathbf{x}) = \nabla^2 u(\mathbf{x})$$

$$= \Delta u(\mathbf{x})$$

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The gradient and the Laplacian

- If $u(t, \mathbf{x})$ is a real-valued function of time and space, the gradient is only with respect to space coordinates

$$\vec{\nabla} u(\mathbf{x},t) = \begin{pmatrix} \frac{\partial}{\partial x_1} u(\mathbf{x},t) \\ \frac{\partial}{\partial x_2} u(\mathbf{x},t) \end{pmatrix} \qquad \nabla^2 u(\mathbf{x},t) = \frac{\partial^2}{\partial x_1^2} u(\mathbf{x},t) + \frac{\partial^2}{\partial x_2^2} u(\mathbf{x},t)$$

$$\vec{\nabla} u(\mathbf{x},t) = \begin{pmatrix} \frac{\partial}{\partial x_1} u(\mathbf{x},t) \\ \frac{\partial}{\partial x_2} u(\mathbf{x},t) \\ \frac{\partial}{\partial x_3} u(\mathbf{x},t) \end{pmatrix} \qquad \nabla^2 u(\mathbf{x},t) = \frac{\partial^2}{\partial x_1^2} u(\mathbf{x},t) + \frac{\partial^2}{\partial x_2^2} u(\mathbf{x},t) + \frac{\partial^2}{\partial x_3^2} u(\mathbf{x},t)$$

In one spatial dimension

- The gradient and Laplacian simplify to the partial derivative and second partial with respect to the one spatial variable if there is only one spatial dimension:

$$\vec{\nabla} u(x,t) = \frac{\partial}{\partial x} u(x,t) \qquad \nabla^2 u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$$

In one spatial dimension

- In this topic, we will approximate solutions to:
  - The heat equation in one spatial dimension
  - The wave equation in one spatial dimension
  - Laplace's equation in two and three spatial dimensions
  - The heat equation and wave equation in two and three spatial dimensions

Approximating solutions to partial differential equations

Summary

- Following this topic, you now
  - Have reviewed the definition of partial derivatives
  - Have seen how to approximate partial derivatives
  - Understand the idea the gradient and Laplacian
  - Are aware of the upcoming topics

References

[1] https://en.wikipedia.org/wiki/Partial_derivative

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Approximating solutions to partial differential equations

Acknowledgments

None so far.

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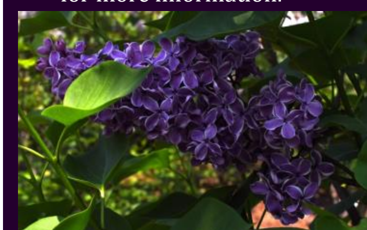
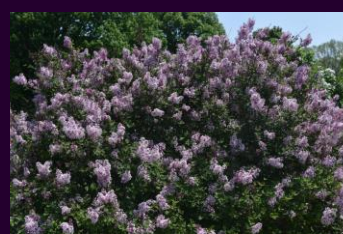
Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using `Consolas`.
Mathematical equations are prepared in MathType by Design Science, Inc.
Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

https://www.rbg.ca/

for more information.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